Practicing

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Algebraic Skills
A Conceptual Approach

Integrating procedures and thinking processes makes learning more meaningful.

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Traditionally, a considerable part of teaching and learning algebra has focused on routine practice and the application of rules, procedures, and techniques. Although today's computerized environments may have decreased the need to master algebraic skills, procedural competence is still a central component in any mathematical activity. However, technological tools have shifted the emphasis from performing operations on complex algebraic expressions to understanding their role and meaning. Consequently, learning rules and procedures should be linked to a deeper understanding of their meaning and to a flexible choice of solution methods (Kieran 2004; Star 2007; NCTM 2000).

We write for mathematics teachers who wish to add a conceptual dimension to the practice of algebraic procedures. Our approach is based on the potential advantages of the traditional approach: Short exercises are readily accessible to students and are easy to implement in regular classrooms, and the focus is on learning one specific skill at a time. The activities we propose here offer opportunities for more effective learning of algebra.

We describe an approach in which rules, procedures, algorithms, sense making, meaningful reading, and the creation of algebraic expressions are thoroughly integrated into the learning process. These practice-oriented activities require the adoption of some additional higher-order thinking skills, such as developing alternative solutions, evaluating the effectiveness of approaches, participating in class discussions, and reflecting on learned procedures and solution methods. The goal is to provide teachers with an alternative to the traditional practice sections of a beginning algebra course without changing the basic format of short exercise sets.

Table 1 presents the framework that guided us in the design of a collection of exercises for beginning algebra students (Resnick, Bouhadana, and Friedlander 2007). The rationale for and the structure of the framework can be summarized in a two-dimensional matrix: One dimension refers to procedures covered by most beginning algebra courses, and the other dimension refers to the cognitive processes involved and required for meaningful learning of these skills.

The components of the cognitive dimension were based on the following sources:

- Research on learning algebra in general and on algebraic procedures in particular (e.g., Mason, Graham, and Johnston-Wilder 2005; Stacey, Chick, and Kendal 2004)

Table 1 Framework

Procedure \ Cognitive Process	Direct Application of Procedures	Reverse Thinking	Global Comprehension	Constructing Examples and Counterexamples	Identifying Errors and Misconceptions	Considering and Justifying Multiple-Choice Tasks	Meaningful Application of Algebraic Operations	Qualitative Thinking	Divergent Thinking
Operations with Negative Numbers									
Manipulating Algebraic Expressions									
Solving Equations and Inequalities									
Solving Systems of Equations									
Factorization									

Sample Task 1

Fill in the missing operation signs. Use parentheses, if needed.

(*a*) $6m \bigcirc 7 \bigcirc 2m = 8m + 7$

(*b*) $6m \bigcirc 7 \bigcirc 2m = 20m$

(*c*) $6m \bigcirc 7 \bigcirc 2m = 12m^2 - 14m$

Sample Task 2

In the following magic squares, the three expressions in each row, column, and diagonal of a square add up to the same magic sum. Fill in the missing expressions.

(*a*)

$x + 1$		
$4x - 1$	$x - 2$	
$-2x - 6$		

(*b*)

$-x$		$\frac{x}{6}$
	$\frac{x}{3}$	
$\frac{x}{2}$		

Fig. 1 In both sample tasks, students are required to perform "reverse" thinking.

- Reported classroom episodes related to algebraic thinking (e.g., NCTM 2000)
- Our own classroom and in-service teaching experience

We will discuss each of the nine processes of the cognitive dimension in algebraic practice. Note that this discussion is not intended to be exhaustive. As mentioned, our format is intended to resemble the traditional approach. Thus, the present collection of tasks does not include elaborate inquiry problems that are longer in scope and richer in the required skills.

THE NINE PROCESSES OF THE COGNITIVE DIMENSION

1. Direct Application of Procedures

The principles and the tasks that correspond to this process are the most common type of exercises found within a traditional approach—for example, requiring students to simplify algebraic expressions or to solve equations. According to our approach, more time is devoted to tasks of the following types.

2. Reverse Thinking

In contrast with simple procedural tasks, the "direction" of the activity is reversed; it calls for backward thinking or for reconstructing a procedure already performed but missing. Reverse thinking requires reconstructing expressions or equations according to given parts or the final result of an exercise (see the two sample tasks shown in **fig. 1**).

Work on the first example in **figure 1** can be followed by a class discussion based on some of the following issues:

- Why are some exercises more difficult than others?
- What kind of reasoning is required in these more difficult cases?

3. Global Comprehension

This ability involves dealing with a multiple-term expression as a single unit (Jensen and Wagner 1981), rather than as viewing it as a collection of many "atomic" components (variables, numbers, and operations). This holistic approach requires a global view and the identification of "units" of reference within complex expressions (see the two

Sample Task 1

Knowing that $2x + 15 = -2$, find the values of the following expressions.

(a) $2x + 16 =$ _____
(b) $2x + 20 =$ _____
(c) $2x + 5 =$ _____
(d) $3 \cdot (2x + 15) =$ _____
(e) $-1 \cdot (2x + 15) =$ _____
(f) $-0.5 \cdot (2x + 5) =$ _____

Sample Task 2

Given that $a - b = 9$ and that $ab = 36$, find the values of the following expressions *without* finding the values of a and b:

(a) $a^2 - 2ab + b^2$ (b) $2ab$ (c) $2a - 2b$

Use your results to find the values for the following:

(d) $a^2 + b^2$ (e) $(a + b)^2$ (f) $a^2 - b^2$

Fig. 2 Students who see a "big picture" complete these exercises more easily.

Complete the quiz about equivalent expressions by writing four reasonable choices for each expression. At least one of the choices should be correct, but more than one may be correct.

Example: $5 - 2 \cdot (7x - 4) =$

(a) $3(7x - 4)$	(b) $-14x + 13$	(c) $5 - 14x + 8$	(d) $5 - 14x - 4$

1. $5 - x - 3 =$

(a)	(b)	(c)	(d)

2. $7 - 2 \cdot (x - 3) =$

(a)	(b)	(c)	(d)

3. $4 \cdot (x \cdot 5) =$

(a)	(b)	(c)	(d)

4. $6 - (2 - x) =$

(a)	(b)	(c)	(d)

Fig. 3 Acting like teachers, students need to create correct and incorrect choices for the "quiz," at least one of which must be correct.

sample tasks shown in **fig. 2**).

In these exercises, students must relate expressions to one another "globally" and identify whole expressions as parts of others. Thus, in the first sample task shown in **figure 2**, exercise (b), students are asked to relate the expression $2x + 20$ to the expression $2x + 15$ (whose value is given as -2). Classroom discussions can include different solutions and comparisons among them; for example, a possible solution is to solve for the unknown x and substitute its value in the other expressions. This strategy is "safe and easy" but also time-consuming, tiring, and error prone. Alternatively, students can opt for global substitutions that require fewer calculations, leading to more efficient solutions, but this approach also requires ingenuity and expertise in algebra.

4. Constructing Examples and Counterexamples

Work on these tasks involves understanding the meaning of a concept or a procedure, applying reverse thinking, justifying, and thinking creatively. This approach also puts students in a "teaching" role; thus, it may induce reflection about a student's own or other students' potential sources of difficulties (see the sample task shown in **fig. 3**).

Individual work on the quiz can be followed by having the class collectively design and agree on a final version. This process involves a discussion of various forms of an algebraic expression and of the most common errors while simplifying a given expression. Perhaps the teacher can give the final version of the quiz to another class and report the results.

Sample Task 1

Is the following solution correct? If not, make the needed corrections.

$$
\begin{aligned}
2(3 - x) + 3(x - 2) + 7x &= 2 \\
6 - 2x + 3x - 6 + 7x &= 2 \\
8x &= 2 \\
x &= 4
\end{aligned}
$$

Sample Task 2

The mathematics teacher asked the class to substitute $a = 3$ in

$$-5 - \frac{3 + 2a}{3}.$$

Check the work of the following students and make the necessary corrections.

Abe $\quad -5 - \frac{3+6}{3} = -5 - 1 + 6 = 0$

Benjamin $\quad -5 - \frac{3+6}{3} = \frac{-15 - 3 + 6}{3} = \frac{-12}{3} = 4$

Claire $\quad -5 - \frac{3+6}{3} = -5 - \frac{9}{3} = -8$

Diane $\quad -5 - \frac{3+6}{3} = -5 - \frac{9}{3} = -2$

Fig. 4 Students need to analyze the work of others to determine which answers are correct and where mistakes lie.

5. Identifying Errors and Misconceptions

In this category, we include the ability to follow, interpret, and evaluate a solution produced by a real or a fictitious peer. As in the previous case, this ability frequently requires justifying, analyzing, and monitoring results as well as thinking critically. These tasks also require reading and reacting to what other students do or may do, a process that may help students look beyond their own knowledge and performance (see the two sample tasks shown in **fig. 4**).

Requiring students to detect errors made by others may raise awareness of their own mistakes. This category of tasks enables teachers to select and discuss the kinds of mistakes they want to address. Thus, the error in sample task 1 in **figure 4** is related to students' tendency to opt for whole numbers and to use direct rather than inverse operations. Following the solution of the substitution task, presented in the second sample task in **figure 4**, the substitution of 3 and –3 in the same expression can be discussed, regarding both the correct solutions and the potential errors.

Which of the expressions below is/are equivalent to

$$\frac{a-6}{3}?$$

$$\frac{a+6}{3} \qquad \frac{1}{3}a-6 \qquad \frac{a}{3}-2 \qquad \frac{6-a}{3} \qquad \frac{1}{3}(a-6) \qquad -\frac{6-a}{3}$$

Which of the expressions below is/are equivalent to

$$\frac{x \bullet y}{3}?$$

$$\frac{x}{3}\bullet y \qquad x\bullet\frac{y}{3} \qquad x\bullet y:3 \qquad \frac{1}{3}(xy) \qquad \frac{x}{3}\bullet\frac{y}{3} \qquad (x\bullet y):3$$

Fig. 5 In this task, students can discuss why each distractor was chosen.

Sample Task 1

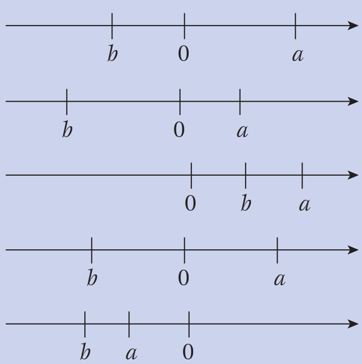
Each rectangle drawn below represents an area of $18x + 36$. Find four possible pairs of expressions that can represent lengths of their sides.

(a) $18x + 36$ (b) $18x + 36$

(c) $18x + 36$ (d) $18x + 36$

Sample Task 2

In this task, a and b represent numbers on the number line. Mark the operations that produce a *negative result* for $a \square b$.



(a) + – × None

(b) + – × None

(c) + – × None

(d) + – × None

(e) + – × None

Fig. 6 Both of these exercises will allow students to review the rules of algebraic operations.

6. Considering and Justifying Multiple-Choice Tasks

In this case, the solution process involves identifying errors, considering multiple solution methods or answers, understanding a concept in depth, and thinking critically (see the sample task shown in **fig. 5**).

7. Meaningful Application of Algebraic Concepts

This goal is addressed explicitly by requiring students to work with multiple representations and multiple solution methods or answers and by discussing solution processes (rather than results), as in the two sample tasks shown in **figure 6**.

Note that a main goal of the two tasks is to stimulate observation of and reflection on rules of algebraic operations. Yet the two exercises pursue this goal differently. Sample task 1 in **figure 6**, presented in the context of area calculations, requires decomposing a given product into two factors in different ways. This process requires reverse thinking and examining several possibilities. The richness of the possible outcomes depends on students'

Sample Task 1

The average of four numbers is negative.

(a) Can all four numbers be negative? Explain.
(b) Can all four numbers be positive? Explain.
(c) Can only two of the four numbers be positive? Explain.
(d) Can only three of the four numbers be negative? Explain.
(e) Can only one of the four numbers be negative? Explain.

Sample Task 2

Without multiplying the factors, predict the number of terms for each simplified product.

(a) $(x + 8)(x - 6)$ (d) $(x + 8)(y - 6)$
(b) $(x + x)(8 - 6)$ (e) $(x + y)(6 - 8)$
(c) $(x + 8)^2$ (f) $(x + 8)(x - 8)$

Fig. 7 These exercises call for students to make predictions or interpret results without using computational skills.

Sample Task 1

Complete equivalent expressions in different ways.

(a) _____ $+ \; 6x \; +$ _____ $= ($ _____ $+$ _____ $)^2$

(b) _____ $+ \; 6x \; +$ _____ $= ($ _____ $+$ _____ $)^2$

(c) _____ $+ \; 6x \; +$ _____ $= ($ _____ $+$ _____ $)^2$

(d) _____ $+ \; 3x \; +$ _____ $= ($ _____ $+$ _____ $)^2$

(e) _____ $+ \; 3x \; +$ _____ $= ($ _____ $+$ _____ $)^2$

Sample Task 2

- Classify the following expressions into two groups. Describe each group.
- Classify the same expressions, this time into three groups. Describe each group.

$x^2 - 8x + 16 \quad x^2 - 16 \quad x^2 + 8x + 16 \quad x^2 + 16$

$x^2 + 10x + 25 \quad x^2 + 25 \quad x^2 - 25 \quad x^2 + 10x - 25$

Fig. 8 Both exercises encourage students to explore solution spaces for a certain problem and to avoid fixation on certain rules.

1. Fill in the missing expressions to complete algebraic identities.

(a) $18x - 36 = 3($ ___ $-$ ___ $)$

(b) $18x - 36 = 6($ ___ $-$ ___ $)$

(c) $18x - 36 = 18($ ___ $-$ ___ $)$

2. Fill in the missing expressions to complete algebraic identities.

(a) $18x - 12x^2 = 3($ ___ $-$ ___ $)$

(b) $18x - 12x^2 = 6($ ___ $-$ ___ $)$

(c) $18x - 12x^2 = x($ ___ $-$ ___ $)$

3. Fill in the missing expressions to complete algebraic identities.

(a) $15x -$ ___ $= 3($ ___ $- 7)$

(b) $8x - 4x^2 =$ ___ $($ ___ $- 4x)$

(c) $18x - 30 =$ ___ $(3x -$ ___ $)$

Fig. 9 These tasks involving thinking backward have different levels of difficulty.

mathematical sophistication—for example, whether one or both factors should contain a variable. A possible follow-up to this example can be collecting and classifying student answers and assessing the more creative solutions.

The second example in **figure 6** relies on the number line as a meaningful visual support for predicting the sign of an operation without actually performing it. In a discussion, students noted that the absolute value of the operands provides a critical piece of information in establishing the sign of a sum or a difference but contributes no knowledge regarding the sign of a product or a ratio.

8. Qualitative Thinking

This ability can be applied in parallel to the formal solution processes of an exercise; it involves predicting, monitoring, and interpreting results. In this case, the addressed manipulation skill is in the background as resource knowledge; it guides students' thinking, but it is not performed (see the two sample tasks given in **fig. 7**).

In the first sample task in **figure 7**, students can either search for examples or formulate a general conclusion. In the first item (a), teachers can discuss the incorrect comment made by some students that a negative average always requires that "all" the numbers that are averaged are negative. By working on the last three items, students will realize that the absolute values of the numbers involved play an important role in considering the propositions.

In discussing the second sample task in **figure 7**, students may realize that they are able to predict some characteristics of a result without performing it, and thus they may develop the ability to do so. This ability is a useful strategy for predicting, monitoring, and checking the validity of their results. Note that the second sample task can also serve as an introduction to learning the multiplication formulas $(a \pm b)^2$ and $(a + b)(a - b)$.

9. Divergent Thinking

This ability is frequently required by tasks that involve and promote multiple solution methods, a wide variety of answers, meaningful mathematical discussions, and opportunities for creative solutions (see the two sample tasks given in **fig. 8**).

These exercises promote divergent thinking at two different levels. Besides incorporating reverse thinking, the first exercise in **figure 8** adds the dimension of "freedom" to create both an expression and its expansion under a constraint (a given middle term for the expansion). The second exercise in **figure 8** requires students to observe, analyze, and group algebraic expressions according to criteria of their choice. In a class discussion of the expansion of a squared binomial (sample task 1 in **fig. 8**), we can address the possibility of having noninteger coefficients as well as odd coefficients for the middle ($2ab$) term.

As noted, some of the nine processes described involve interrelated abilities and overlap to a certain extent. Thus, we propose that the framework be considered more a list of task characteristics and thinking processes than a clear-cut classification system of algebraic exercises. This framework serves as a guide for selecting or designing exercises that promote meaningful learning of procedural activities in algebra.

LEVELS OF DIFFICULTY

Our framework is based on the assumption that all students are capable of engaging in the cognitive processes described above, provided that some adaptations are made.

The difficulty level of an algebraic exercise set is usually determined by the following criteria:

- *Level of technical difficulty*—Determined by the representation or the size of the involved numbers (large versus small numbers, integers, decimals or fractions) and by task complexity (number of operations and processes to be performed in parallel)
- *Level of acquaintance with the content and the solution method*—Determined by the degree of similarity to the content or the solution method of other previously encountered tasks

Consider the sample task shown in **figure 9**. The three exercise sets in this example relate to the same procedure—factorization of algebraic expressions by applying the distributive property. The sets require the same strategy (reverse thinking) but have different difficulty levels. The first set contains nonnegative integers and linear expressions and, as a result, has a lower technical difficulty level. The location of the blanks for missing expressions within an exercise provides an additional aspect of these tasks' level of difficulty. For example, sample task 3 is more complex compared with the exercises in the other sets.

CONCLUSION

The framework and the tasks previously discussed integrate conceptual components into the learning and practicing of algebraic procedures.

We borrowed the following from the traditional perspective:

- Algebraic skills are an important component of algebra and should be explicitly practiced.
- Short exercises are important because they support achieving clear goals and can provide a sense of accomplishment without long-term investments.
- In each unit, tasks are arranged according to levels of increasing complexity to accommodate students with different abilities.

From a conceptual perspective, we attempted to—

- increase the cognitive demand by encouraging students to use a variety of thinking processes, strategies, and reflective practices;
- promote divergent thinking and creativity, such as requiring students to design exercises themselves; and
- create opportunities for meaningful discussions of algebraic skills that go beyond teacher feedback about whether a response is correct or not.

Our approach attempts to maintain an appropriate balance between the various types of cognitive demands on students and the skills that they need to learn. Our initial observations of students' work indicate that, in addition to improving their procedural knowledge, these exercises provide opportunities for students to promote meaningful learning and increase their motivation.

REFERENCES

Jensen, Robert, and Sigrid Wagner. 1981. "Three Perspectives on the Process Uniformity of Beginning Algebra Students." In *Proceedings of the Second Annual Conference of the North American Chapter of the International Group for the Psychology of Mathematics Education* (*PME-NA*), edited by Sigrid Wagner, pp. 133–39. Athens, GA.

Kieran, Carolyn. 2004. "The Core of Algebra: Reflections on Its Main Activities." In *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study*, edited by Kaye Stacey, Helen Chick, and Margaret Kendal, pp. 21–34. Dordrecht, The Netherlands: Kluwer.

Mason, John, Alan Graham, and Sue Johnston-Wilder. 2005. *Developing Thinking in Algebra*. London: The Open University.

National Council of Teachers of Mathematics (NCTM). 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Resnick, Zippora, Rachel Bouhadana, and Alex Friedlander. 2007. "Algebraic Skills for Grades 7–8: A Collection of Tasks." CD and website (in Hebrew). Rehovot, Israel: The Weizmann Institute of Science. http://multimedia.haifa.ac.il/algebra_cd.

Stacey, Kaye, Helen Chick, and Margaret Kendal, eds. 2004. *The Future of the Teaching and Learning of Algebra: The 12th ICMI Study*. Dordrecht, The Netherlands: Kluwer.

Star, Jon R. 2007. "Foregrounding Procedural Knowledge." *Journal of Research in Mathematics Education* 38 (March): 132–35.

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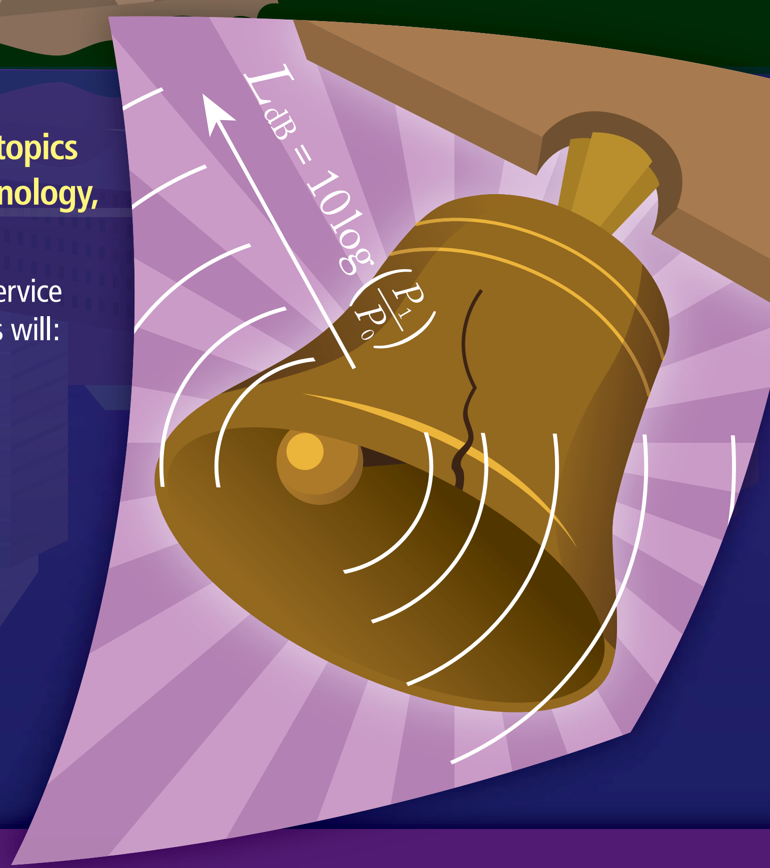

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