Chapter 4

Examining Mathematics Learning from a Conceptual Change Point of View: Implications for the Design of Learning Environments

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The Problem of Knowledge in the Design of Learning Environments

Most educational researchers seem to agree that mathematics learning does not consist of the passive absorption of certain abstract, de-contextualized concepts and procedural skills to be acquired by individuals through transmission teaching methods. Rather, researchers talk about the development of a mathematical disposition which involves not only domain-specific knowledge and problem-solving skills, but also meta-knowledge, self-regulatory skills, motivational factors, and epistemological beliefs about mathematics (e.g., De Corte, Greer, & Verschaffel, 1996; Schoenfeld, 1992, 2002). Nevertheless, the discussion about the design of powerful learning environments that can foster the development of a mathematical disposition is not yet settled, often reflecting the controversy between cognitive and situated approaches to learning and teaching.

The situated approach movement has drawn the attention of the mathematics education community to certain aspects of learning that were not considered important in the context of cognitive approaches, such as the relevance of the social and cultural context in which learning takes place, and the role of artefacts in learning (Anderson, Reder, & Simon, 1996; De Corte, 2004; Sfard, 1998). The situated approach has also been useful in pointing out the mismatch between the way mathematics is taught in the schools and the way it is used in real-life situations. It is argued that many school activities may be meaningless for students and this may be a source of creating inert knowledge that cannot be transferred

[1] The present study was funded through the program EPEAEK II in the framework of the project "Pythagoras - Support of University Research Groups" with 75% from European Social Funds and 25% from National Funds.

Instructional Psychology: Past, Present, and Future Trends: Sixteen Essays in honour of Erik De Conte

ISBN: 0-08-045021-0

to out-of-school situations (Brown, Collins, & Duguid, 1989). Thus, it has been suggested that training by abstraction is of little use, that instruction needs to be integrated in complex, social environments, and that learning should take place in authentic contexts.

Although the situated approach has served an important role in de-emphasizing individual, cognitive — 'in the head' — internal processes and stressing instead the social, collaborative, and realistic aspects of learning that were previously neglected, it has also led to some misguided claims and inappropriate educational suggestions, as pointed out by Anderson et al. (1996). For instance, the claims that learning is always grounded in concrete situations, that knowledge does not transfer between tasks, and that training by abstraction is of little use seem to draw only on limited empirical data and to neglect findings that they cannot predict. Another important limitation of the situated approach is that it has downgraded the problem of knowledge (Bereiter, 1997; Vosniadou, 2005). This is the case, because according to situativity theory, knowledge is not a product to be acquired but a process in which one participates — the well-known debate between the 'acquisition' and 'participation' views on knowledge and learning.

If we see knowledge as a process, then the emphasis on teaching shifts from teaching subject-matter content to teaching thinking and learning skills. Indeed, the main metaphor developed within the situated cognition approach has been the cognitive apprenticeship metaphor (Collins, Brown, & Newman, 1989). The problem with the cognitive apprenticeship metaphor is that it has not solved sufficiently the problem of finding an authentic culture for students to participate in expert practice. As argued in Vosniadou (2005), we can neither expect to enculturate primary and secondary school students in the cultures of the mathematicians, physicists, historians, etc., nor expect to produce 'intelligent novices' who are trained in expert practices ignoring the bodies of knowledge and systems of beliefs that go together with developing expertise in a given domain of knowledge. Cognitive apprenticeship becomes empty when its purpose is to practice cognitive skills in the absence of substantial knowledge building and where the domain knowledge is secondary to the learning skills that are to be acquired. According to Bereiter (1997), situativity theory has failed to provide a cogent idea of schooling and a new educational vision exactly because it has not been able to adequately address the problem of knowledge.

Science and mathematics are not only processes in which one participates, but also the knowledge products of a complex social interaction. Although they are produced through the situated practices of scientists, they nevertheless have an objective non-situated reality, which is divorced from the processes that produced them. Just like paint is produced through a situated manufacturing process but as a product it can be used in many different situations, and thus it is not in itself situated (see Bereiter, 1997; Vosniadou, 2005), so is the case with scientific and mathematical knowledge. It can be used by engineers and architects, by computer scientists and artists, in situations very different from those that produced it, to create new knowledge and artefacts that may change the very culture in which we live.

This point does not weaken the situated cognition approach thesis. Rather, it calls for an integrated approach to the design of learning environments, striving not only for the acquisition of mathematical skills through cognitive apprenticeship but also for deep, conceptual, subject-matter knowledge. The conceptual change theoretical framework, which will be described in detail below, addresses aspects of the problem of knowledge in the design of learning environments for the purpose of developing deep understanding of subject-matter knowledge.

The Conceptual Change Theoretical Framework

The conceptual change approach to learning and teaching has its roots in Thomas Kuhn's revolutionary account of theory change in the history of science with the publication of his book 'The structure of scientific revolutions' (Kuhn, 1970). In this book Kuhn argued that scientific theories develop in the context of paradigms — i.e., webs of shared concepts, beliefs, practices — and shared commitments that students of science learn when they become scientists, and that theory change is not a cumulative but a revolutionary process during which the old paradigm is rejected and replaced with a new one. This account of theory change has served as a source of hypotheses about how concepts change not only in the philosophy and history of science but also in the process of learning science (Posner, Strike, Hewson, & Gertzog, 1982).

The basis for the analogy between scientific theory change and the learning of science became the realization that students bring to the science learning task 'preconceptions', 'misconceptions', or 'alternative conceptions' (Driver & Easley, 1978; Novak, 1977; Viennot, 1979) that stand in the way of learning science. Posner et al. (1982) and also McCloskey (1983) argued that these alternative conceptions can be seen as theories that need to be replaced by the currently accepted, correct, scientific views through a process of conceptual change. For this conceptual change to be achieved students need to experience dissatisfaction with their existing ideas, a dissatisfaction usually produced in instructional settings through cognitive conflict, and must understand the fruitfulness of the new, scientific explanations.

The conceptual change approach was the leading paradigm in science education until it became subject to several criticisms, regarding both its epistemological assumptions and its instructional practices. Among other things, it has been pointed out that this theoretical framework provides a rather simplistic view of misconceptions, as being unitary, faulty conceptions, and that it ignores their interrelations with other concepts, as well their interaction with the situational context in which they are invoked (Caravita & Halldén, 1994; Smith, diSessa, & Rochelle, 1993). This view of misconceptions underlies the instructional practice of cognitive conflict that emerged from this theoretical framework, which aims at replacing students' misconceptions with correct ideas. This practice has been criticized as not having a sound constructivist basis, as ignoring students' productive ideas, and therefore, as being misguided and inefficient (Smith et al., 1993). In addition, Caravita and Halldén (1994) pointed out that conceptual change happens in a larger situational, educational, and socio/cultural context; that it is affected by motivational and affective variables; and that one cannot ignore the fact that science is socially constructed and validated (see also Driver, Asoko, Leach, Mortimer, & Scott, 1994; Pintrich, 1999).

In the meantime, cognitive-developmental psychologists have also started investigating the conceptual change framework as a source of ideas to explain how concepts develop in the growing child (Carey, 1985; Vosniadou & Brewer, 1987). Research with infants has shown that children interpret their everyday experience in order to form broad explanatory frameworks that commit them to an ontology and causality that distinguishes physical from psychological objects and which forms the basis for the knowledge acquisition process. For example, infants soon after birth seem to construct certain principles regarding the behaviour of physical objects (such as that they are solid, stable; that they do not move by themselves; that they fall 'down' when unsupported, etc.) that make it possible for them to participate in the physical world (i.e., Baillargeon, 1994; Spelke, 1991). This initial framework theory usually facilitates the learning of science through a process of continuous enrichment, as new information is added to the existing conceptual structures. However, when the new information to be acquired is radically different than what is already known (i.e., different in its structure, in the phenomena it explains, as well as in the very concepts that comprise it), prior knowledge may hinder the acquisition of the new, intended knowledge and the process of enrichment can result in the production of misconceptions (Gelman & Meck, 1992; Vosniadou & Verschaffel, 2004).

Vosniadou and her colleagues have attempted to provide detailed descriptions of the development of knowledge in several areas of the natural sciences, such as observational astronomy (Vosniadou, 1994, 2003; Vosniadou & Brewer, 1992, 1994), mechanics (Ioannides & Vosniadou, 2001; Megalakaki, Ioannides, Vosniadou, & Tiberghien, 1997), geophysics (Ioannidou & Vosniadou, 2001), chemistry (Kouka, Vosniadou, & Tsaparlis, 2001), and biology (Kyrkos & Vosniadou, 1997). The results of these studies have shown that young children answer questions about force, matter, the earth in space, or about the composition of earth, mostly in an internally consistent way, revealing the existence of narrow but coherent initial explanatory frameworks. In the process of learning science, children usually add the new, scientific, information to their initial explanatory frameworks. The framework theory approach to conceptual change predicts that new information which is incompatible with what is already known is more difficult and time consuming to be learned than new information that can enrich existing structures. Moreover, given that learners are usually not aware of the ontological and epistemological commitments of their initial framework theory, they are most likely to use the same additive mechanisms with all forms of new knowledge making likely the formation of misconceptions. Many misconceptions can be explained as *synthetic models* reflecting students' attempts to assimilate new information in their existing but incompatible knowledge structures. In the case of observational astronomy, examples of such synthetic models are the model of the dual sphere, the hollow sphere, or the flattened sphere; the model of the sun and the moon revolving around a spherical earth in a geocentric solar system, etc. (see Vosniadou & Brewer, 1992, 1994).

The framework theory approach to conceptual change that we adopt meets all the criticisms of Caravita and Halldén (1994) and Smith et al. (1993). First, misconceptions are not considered as unitary, faulty conceptions. Rather, we describe a knowledge system consisting of many different elements organized in complex ways. Second, we make a distinction between the learner's initial explanatory framework, prior to systematic instruction, and misconceptions that are produced after instruction. We explain the formation of the initial framework theories taking into consideration evolutionary factors, as well as young children's interaction with their physical and social environment and the cultural tools which they have available. Third, our theoretical position is a constructivist one. Not only it assumes that new information is built on existing knowledge structures, it also uses constructivism to explain students' misconceptions and to provide a comprehensive framework for making meaningful and detailed predictions about the knowledge acquisition process. More specifically, according to the conceptual change point of view that we propose, many misconceptions are formed precisely because learners have the tendency to enrich their prior knowledge with new information even when this information is totally incompatible with what they already know. Finally, while our approach investigates mainly the cognitive facets of conceptual change, it is complementary and not contradictory to other approaches that deal with motivational/affective and socio/cultural factors (Anderson, Greeno, Reder, & Simon, 2000).

The Framework Theory Approach to Conceptual Change and the Acquisition of Mathematical Knowledge

It has been argued that the framework theory approach to conceptual change can be fruitfully applied in the case of mathematics learning (Vosniadou & Verschaffel, 2004). As it is the case that students develop an initial framework theory about physics on the basis of everyday experience, they also develop an initial framework theory about number, organized on the basis of certain core principles or presuppositions. Indeed, there is growing evidence that, even before any instruction, children form an initial understanding of number, based on their experience with natural numbers. According to Gelman (2000), children construct a 'principled understanding' of numbers on the basis of the act of counting.

In science learning, students' framework theories facilitate some kinds of learning but inhibit others. There is evidence that this also happens in the case of mathematics learning. For example, an early understanding of natural number and its properties supports children's understanding of notions such as potential infinity (Hartnett & Gelman, 1998), while at the same time it stands in the way of students' understanding of the properties and operations of rational numbers (Carpenter, Fennema, & Romberg, 1993; Moskal & Magone, 2000; Resnick et al., 1989; Yujing & Yong-Di, 2005).

These findings support the argument that science and mathematics learning share certain similarities in terms of the significance of students' initial explanatory frameworks for the acquisition of new, intended knowledge. A number of empirical studies conducted within the conceptual change theoretical framework have been presented in a special issue of *Learning and Instruction* (Verschaffel & Vosniadou, 2004). The research presented in this special issue has shown that the framework theory approach to conceptual change can be fruitfully applied to predict and explain students' difficulties with fractions (Stafylidou & Vosniadou, 2004), the use of the negative sign in algebra (Vlassis, 2004), the number concept (Merenluonto & Lehtinen, 2004; Vamvakoussi & Vosniadou, 2004b), as well as students' tendency to apply linear models in cases where they are not applicable (Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2004).

In the following section, we will present in greater detail two studies conducted in our lab which apply the framework theory approach to conceptual change to explain students' difficulties with rational numbers.

Developing the Concept of Rational Number

As we have already mentioned, even before instruction, children construct a framework theory of number on the basis of the principles of counting. This foundational theory basically treats all numbers as natural numbers.

Analysing the differences between the set of natural and the set of rational numbers, we point out two major differences: First, the set of natural numbers is discrete, whereas the set of rational numbers is dense. In the set of natural numbers, there is a finite number of numbers between any two numbers, whereas in the set of rational numbers there are infinitely many numbers between any two, non-equal, numbers. Second, any number within the set of natural numbers has a unique symbolic representation, whereas any number in the set of rational numbers has multiple symbolic representations; e.g., the symbols 4/2, 32/16, 2.0, etc., are different representations of the same numerical value 2.

These particular differences between the set of natural and the set of rational numbers suggest that understanding about the structure of the set of rational numbers may be difficult for students. Indeed, there is evidence that the idea of discreteness is a barrier to understanding about density for students (Malara, 2001; Merenluoto & Lehtinen, 2002, 2004; Neumann, 1998), as well as for elementary school prospective teachers (Tirosh, Fischbein, Graeber, & Wilson, 1999). There is also evidence that the symbolic representation of the numbers involved may be a factor that interferes with students' understanding about the structure of rational numbers intervals. For instance, Neumann (1998) reports that 7th graders had difficulties accepting that there could be a fraction between 0.3 and 0.6, probably because they thought of decimals and fractions as different, unrelated sorts of numbers. There is yet another way through which symbolic notation may affect students' thinking about numbers: Students may consider decimals, as well as fractions, as different, unrelated subsets of the set of rational numbers. This assumption is supported by research in various domains providing evidence that novices in a domain tend to group objects on the basis of superficial characteristics (e.g., Chi, Feltovich, & Glaser, 1981). This disposition may be enhanced by the fact that the operations as well as the ordering of fractions are considerably different than those of decimals.

We claim that understanding about the dense structure of the rational numbers set requires conceptual change. On the basis of the framework theory approach to conceptual change described by Vosniadou (1994, 2001, 2003), we assume that the development of the concept of rational number is a slow and gradual process, constrained by the presupposition of discreteness as well as by the presupposition that any number has a distinct symbolic representation. Therefore, we expect that there will be intermediate levels of understanding of the concept of rational number and that students will generate *synthetic models*, reflecting the idea of discreteness and the belief that different symbolic representations refer to different numbers.

These hypotheses were tested in two studies. In the first (Vamvakoussi & Vosniadou, 2004b), 16 9th graders participated in a 45-min individual interview during which they were asked to define the number of numbers between two rational numbers: A pair of decimals with the same number of decimal digits, a pair of decimals with different number of decimal digits, a pair of similar fractions, a pair of dissimilar fractions, a fraction and a decimal. We found that the majority of our participants (11 out of 16) answered that there is finite number of numbers in all questions, describing intervals that preserve the discrete structure of natural numbers. Only 1 out of 16 students answered that there are infinitely many numbers both in the case of decimals and in the case of fractions. Even this student was reluctant to answer that there are infinitely many numbers between a decimal and a fraction, although he was informed that the particular numbers were not equal; instead, he explicitly said that he needed to turn the fraction into a decimal first, indicating that the symbolic notation of numbers constrained his understanding of density.

The same constraint appeared in the responses of the remaining five students, who answered that there are infinitely many numbers in some but not in all questions. Two of these students changed their answer according to the symbolic representation of the numbers involved. For example, one of them answered that there is a finite number of numbers between decimals, but there are infinitely many numbers between fractions. He also explained that if one turns the decimals into fractions, one can find infinitely many numbers in between, indicating his belief that the structure of the interval differs, according to the symbolic representation of the first and the last number, even when the corresponding numerical values do not change.

In addition, two of the participants explicitly expressed the belief that different symbolic representations refer to different numbers. For example, one of them answered that there is finite number of numbers in all questions, but she answered that there are infinitely many numbers between 3/8 and 5/8, mentioning several different symbolic representations of the fraction 4/8.

As it is apparent, the above results support our hypotheses (a) that students find it difficult to understand the concept of rational number, (b) that the areas of difficulty are exactly those aspects of rational numbers that are inconsistent with the presuppositions of natural numbers (discreteness and unique symbolic representation), and (c) that students create intermediate conceptions (misconceptions, errors, synthetic models) that reveal their tendency to assimilate aspects of the new concept to their incompatible knowledge base.

A second study (Vamvakoussi & Vosniadou, 2004a, in press) followed, with the purpose of further investigating the effect of the idea of discreteness and of the symbolic notation of numbers on children's understanding of the structure of the rational numbers set. We were also interested to test the effect of an external representation of real numbers, namely the number line, on children's responses to tasks regarding density. Following an ongoing discussion in the conceptual change literature about the effect of external representations on students' reasoning and understanding (Vosniadou, Skopeliti, & Ikospentaki, 2005), we assumed that the effect of the number line is rather limited and that it may disappear when the number line is withdrawn.

The participants of the second study were 301 students, 164 9th and 137 11th graders. We designed two types of questionnaires, one open-ended and one forced-choice questionnaire, which consisted of items regarding the number of numbers between two rational numbers, as well as their symbolic representation. The number line was present in half of the items. The numbers 0 and 1 as well as the first number of each given interval were already placed on the number line. Our participants completed the questionnaires in two phases: Half of them received the questions with the number line at the first phase, whereas the other half received the questions with the number line at the second phase. In both cases, the first part of the questionnaire was withdrawn before the second part was handed out.

Here we will refer only to the results obtained in the forced-choice condition. An example of the items without and with the number line is shown in Table 1. In this example, the answers are ordered from the more naïve to the more sophisticated (i–iv). Students received the answers in random order.

The results showed that the presupposition of discreteness was still strong in the 11th grade, although the performance of the 11th graders was significantly better than the performance of the 9th graders. Indeed, about one third of the 11th graders answered that there is finite number of numbers in each of the items of the questionnaire. On the basis of participants' responses in all the items, we categorized them in five categories. Table 2 presents the categorization and the corresponding percentages.

With the exception of those placed in the 'Density' category, students gave alternative accounts of the structure of the rational numbers intervals, reflecting the presupposition of discreteness and the effect of the symbolic notation of numbers. Students in the 'Discreteness' category treated all given numbers as if they were successive — e.g., they answered that there are no numbers between 0.005 and 0.006. Students in the 'Refined Discreteness' category did not consider the given numbers to be successive but still answered that there is finite number of numbers between the given numbers — for instance,

Table 1: Second study: Examples of the items in the forced-choice questionnaires (Vamvakoussi & Vosniadou, 2004a, in press).

Without the number line
• **How many numbers are there that are greater than 0.005 and, at the same time, less than 0.006?** **i.** There is no such number. **ii.** There are the following numbers: 0.0051, 0.0052, 0.0053, 0.0054, 0.0055, 0.0056, 0.0057, 0.0058, 0.0059. **iii.** There are infinitely many decimals. **iv.** There are infinitely many numbers: simple decimals, decimals with infinitely many decimal digits, fractions, square roots. **v.** None of the above. I believe that ….
With the number line
• **Place 0.2 on the number line.** **How many numbers are there that are greater than 0.1 and, at the same time, less than 0.2?** **i.** There is no such number. **ii.** There are the following numbers: 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19. **iii.** There are infinitely many decimals. **iv**. There are infinitely many numbers: simple decimals, decimals with infinitely many decimal digits, fractions, square roots. **v.** None of the above. I believe that ….

Table 2: Second study: Categorization of the participants (Vamvakoussi & Vosniadou, 2004a, in press).

	Forced-choice questionnaires	
Category	**9th graders (*N*=81)**	**11th graders (*N*=71)**
Discreteness	4.9%	4.2%
Refined discreteness	30.9%	16.9%
Mixed	40.7%	42.3%
Constrained density	12.3%	15.5%
Density	11.1%	21.1%

that between 0.005 and 0.006 there are only the numbers 0.0051, 0.0052, … , 0.0059. Students in the 'Mixed' category answered that there are infinitely many numbers in some, but not all questions. More than half of the students in this category responded differently to intervals with different structures, according to the symbolic representation of the first and the last number. These students answered differently when the first and last numbers of the interval were decimals, as compared to when these numbers were fractions. Students in the 'Constrained Density' category answered that there are infinitely many numbers of the same symbolic representation in the given interval for at least one out of the six questions. These students were reluctant to accept that there may be, for instance, fractions between two decimals. Finally, students in the 'Density' category answered that there are infinitely many numbers, regardless of their symbolic representation, in all the questions.

In addition, 15% of the 11th graders and 19.70% of the 9th graders answered that there are infinitely many fractions between 3/8 and 5/8, all equivalent to 4/8, indicating their belief that different symbolic representations of the number 4/8 count as different numbers.

The effect of the number line on students' performance was quite limited. There was no significant difference in the 11th graders' performance in the questions with and without the number line. The number line improved 9th graders' performance only in one specific case, which involved a shift from hundredths to thousandths. In addition, there was no significant difference in the performance of those who answered the questions with the number line in the first phase compared to those who answered the same questions in the second phase, indicating that the effect of the number line disappeared when the number line was not present. We should also note that it is not the case that the presence of the number line always had a positive effect — in fact, some students performed worse in the questions with the number line.

The results of the second study further strengthened our previous results in showing (a) that the idea of discreteness is strong, both in the case of 9th and 11th graders, (b) that there are intermediate levels of understanding of the concept of density, and (c) that the symbolic representation of the numbers affects students' thinking about the rational numbers' intervals. It seems that students tend to think of fractions and decimals as unrelated mathematical objects, even though they have been explicitly taught how to turn decimals into fractions and vice versa.

The above results are compatible with the predictions coming from the framework theory approach to conceptual change, and therefore support our hypothesis that new information about rational numbers cannot be simply added to what students already know, but requires a radical re-organization of the concept of number. How is such re-organization accomplished and what does it imply for the design of learning environments for teaching and learning mathematics?

Implications for the Design of Learning Environments in Mathematics

De Corte (2004) has outlined certain principles that can be used as a guide for the design of powerful environments for learning mathematics, such as that learning environments should:

- initiate and support active, constructive knowledge acquisition processes in all students;
- allow for students to acquire control over their own learning;
- provide students with the opportunity to elaborate mathematical knowledge in contexts that are meaningful for them;
- create a classroom culture that supports collaboration among students and allows for students to reflect on their learning activities and their epistemological beliefs about mathematics and mathematics learning;
- provide students with the opportunity to build substantial domain-specific knowledge and, at the same time, develop general learning and thinking skills embedded in the subject-matter knowledge.

We fully agree with the above principles, which take into consideration cognitive, metacognitive, affective, and contextual aspects of learning. Focusing on the issue of conceptual change, we will further elaborate on certain guidelines which could be useful for curricula designers, teachers, and researchers who are interested in implementing powerful learning environments for mathematics learning.

Breadth of Coverage of the Curriculum-time Considerations

Conceptual change research shows that, similar to the case of science, the understanding of certain mathematical concepts is a difficult and time-consuming process. This finding calls for a reconsideration of decisions regarding the breadth of coverage of the curricula in mathematics education. It may be more profitable to design curricula that focus on the deep exploration and understanding of certain key concepts and their association with other concepts, both within and outside mathematics, instead of covering a great deal of material in a superficial way, which is likely to lead students to logical incoherence and the creation of misconceptions. In addition, teachers should be aware of the fact that even the most carefully designed learning environment should not be expected to produce results if they do not invest a considerable amount of time in the implementation (see, e.g., Van Dooren et al., 2004).

Order of Acquisition of the Concepts Involved

Usually the concepts that comprise a subject-matter area have a relational structure that influences their order of acquisition. In mathematics education it is often the case that the subject matter is assumed to have a hierarchical structure, in which new concepts follow logically from prior ones. For instance, Grossman and Stodolsky (1995) found that mathematics educators, compared with those of other studies, such as sciences and social studies, consider their subject to be highly sequential. This view of mathematics is compatible with the belief that learning is additive and may hinder mathematics teachers, as well as designers of mathematics curricula, from recognizing that the conceptual change issue is relevant in mathematics learning. Usually, this results in teaching ‘simple’ concepts first and then introducing the more ‘complex’ concepts, which are often presented either as being logical implications of what is already taught or as expansions of prior concepts. For instance, fractions are introduced much later than natural numbers and their operations; algebraic concepts, such as the concept of variable, are taught in secondary education or at the end of elementary school; the set of rational numbers is presented as an expansion of the set of natural numbers; the set of real numbers is similarly introduced as an expansion of the set of rational numbers.

It is important to note that the concepts that are considered to be ‘simpler’ are usually the ones closer to children’s intuitive theories. Thus, children’s initial theories are confirmed and strengthened through instruction, resulting in cognitive inflexibility that hinders further understanding. Furthermore, introducing new concepts as expansions of prior ones is ineffective when what is needed is the restructuring of prior knowledge. There are two suggestions that should be taken into consideration: First, we should consider whether curricula should support the introduction of certain concepts at an earlier stage in mathematics education. For example, there is evidence to support that elaborating notions relative to fractions at an earlier stage in instruction may weaken the effect of early understandings about natural numbers (Yujing & Yong-Di, 2005). There is also an attempt to introduce algebraic thinking in elementary school (Carraher, Schliemann, & Brizuela, 2001). Of course, detailed empirical research is needed before we can introduce such innovations in schools.

Second, we should take into consideration that an expansion of a concept from a mathematical point of view may not correspond to an enrichment of prior conceptual structures. To use an example from our study about the set of rational numbers, students cannot develop their understanding of the set of rational numbers if they do not abandon the presupposition of discreteness, if they do not realize that different symbolic representations may refer to the same mathematical object, and if they do not realize the relation between the ‘set’ of fractions and the ‘set’ of decimals. We suggest that the design of curricula needs to be based on the results of detailed empirical investigations that provide information about the way students develop understanding in a given domain and the order of acquisition of the concepts involved. The conceptual change theoretical framework can be used as a guide for researchers to identify mathematical concepts that are going to cause students difficulty.

Addressing Students’ Initial Theories and Their Entrenched Presuppositions

Special attention must be paid so that the information included in the curricula, books, and that communicated by the teachers does not foster students’ misconceptions and that it addresses students' entrenched presuppositions on the basis of their initial number theory. To state an example from the current Greek 7th grade mathematics book, the rational numbers set is introduced in the following way: 'All the numbers that we know, namely the natural numbers, decimals and fractions, together with the respective negative numbers, constitute the set of rational numbers'. This 'definition' certainly builds on students' prior knowledge about numbers, but at the same time, enhances students' initial tendency to group numbers on the basis of their symbolic representations. In order to avoid the creation of misconceptions, it is essential that designers of curricula and teachers be informed about the issue of conceptual change, as well as about the learning difficulties that this theoretical framework predicts. Towards this direction, a detailed description of the development of certain key concepts should be provided by researchers.

Increasing Students' Metaconceptual Awareness: Opportunities to Express and Elaborate Opinions

Students are usually not aware of their explanatory frameworks and of the presuppositions that constrain them. This lack of metaconceptual awareness prevents students from questioning their prior knowledge and allows for the assimilation of new information into existing conceptual structures. This kind of learning by assimilation seems to form the basis for the creation of misconceptions and lies at the root of the inconsistency so commonly observed in students' reasoning. A powerful learning environment should aim at increasing students' metaconceptual awareness by providing the opportunity for them to externalize their beliefs and make them subject to evaluation by themselves, their fellow students, and their teachers. This can be done in environments that facilitate group discussion and the verbal expression and elaboration of ideas.

Use of External Representations and Cultural Artefacts

The use of manipulatives, models, and cultural artefacts is considered a significant component of powerful learning environments. However, it should be taken into consideration that the mere presence of such tools is not enough to mediate effective learning. Research in mathematics education suggests that the meaning of a mathematical idea is not necessarily carried by a more concrete representation (*Clements & McMillen, 1996)*. It has also been observed that certain components of a manipulative aid or a representation may constrain children's thinking about the concepts involved (see, e.g., Behr & Post, 1981). Moreover, findings from the conceptual change research suggest that an external representation is itself interpreted on the basis of students' prior knowledge (Vosniadou et al., 2005). Our finding that the number line has limited effect on students' performance in questions regarding density can be explained by the suggestion that the metaphor 'numbers are points on a line' is not necessarily easy to comprehend (Lakoff & Nunez, 2000; Nunez & Lakoff, 1998). It may also be the case that students' responses in the questions with the number line reflect their 'theories' about numbers. But it is also possible that the number line, as a representational tool, may pose its own constraints to students' understanding about numbers and, more specifically, about density. For instance, it is possible that some students consider the number line to be subject to constraints pertaining to a real-world object. Or, that the number line itself is interpreted on the basis of the fundamental presupposition of discreteness, as consisting of discrete points. So, barriers in the effectiveness of the number line could arise from students' 'theories' about numbers and/or their initial ideas about the number line itself. Whatever the case, it should be taken into consideration that the use of external representations and artefacts should be supported by explicit teaching and explanations. Teachers should also be aware of the advantages and limitations of the tools they chose to use.

To summarize, the conceptual change theoretical framework may be used as a guide to identify concepts in mathematics that are going to cause students great difficulty, to predict and explain students' systematic errors and misconceptions, to provide student-centred explanations of counter-intuitive math concepts, to alert students against the use of additive mechanisms in these cases, to find the appropriate bridging analogies, etc. In a more general fashion, it highlights the importance of developing students who are intentional learners and have developed the metacognitive skills required to overcome the barriers imposed by their prior knowledge (Schoenfeld, 1987; Vosniadou, 2003).

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